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THE MAXIMUM DRAWDOWN OF DISCRETE TIME PROCESSES

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Abstract. The maximum drawdown (MDD) is a well-known risk measure extensively used in financial markets. It measures the maximum loss from peak to subsequent valley for a stochastic process. In this work we consider discrete time processes, and derive the probability density of the maximum drawdown in terms of integral equation recursions. This is one of the few works that tackle the MDD of discrete time processes, as most of the literature focused on continuous processes.

Keywords: maximum drawdown, discrete stochastic processes, market risk, Fokker Planck.
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1 Introduction

The maximum drawdown (MDD) is one of the major risk measures for stock market investments. It measures the maximum loss from peak to subsequent trough (Burghardt & Duncan, 2003) (see Figure 1 for an illustration). The maximum drawdown is closely watched by investors, because once a portfolio incurs a large drawdown, it could signify a shift in the profitability of the portfolio. To distinguish between a pure chance event and a systemic change to the profitability of the portfolio, one must test the maximum drawdown against a null hypothesis. This means a probabilistic analysis of the maximum drawdown phenomenon is necessary. The maximum drawdown as a mathematical problem applied to a stochastic process has been an active problem since the 1940's, where it was applied to water damn drawdowns. In fact it is considered by some researchers to be one of two most important problems in the application aspects of the theory of stochastic processes (together with the first passage time problem (Atiya & Metwally, 2005)). Most of the work addressed continuous stochastic processes. In this paper we consider discrete time stochastic processes, and derive the probability distribution for the maximum drawdown. The derived formulas are in the form of integral equations. These formulas apply to any given process transition density, and are not restricted to the normal case. We hope this contribution will shed some theoretical insight into the MDD of discrete processes, and will spur some more work on this vital problem.

2 Related Work

The maximum drawdown, since its formulation in the 1940's, has had little progress until the early 2000's. Subsequently Douady et al., (2000) tackled the zero-drift Brownian motion case. Magdon-Ismail et al. (2004), Magdon-Ismail et al. (2003), and Magdon-Ismail & Atiya (2004) derived a formula for the expectation of the MDD for the Brownian motion with drift, in the form of a series expansion. They also analyzed the behavior asymptotically (as time goes to infinity), showing that it undergoes a "double jump" phase transition, from linear to square root to logarithmic, as the drift goes respectively from negative to zero to positive.

Subsequently, more work appeared on the MDD problem. Pospisil & Vecer (2008) computed the expected value of the maximum drawdown by converting the problem to a partial differential equation (PDE) formulation. Zhang & Hadjiliadis (2010), and Hadjiliadis & Vecer (2006) derived the probability that a rally of a units precedes a drawdown of equal units in a random walk model and its continuous equivalent, a Brownian motion model in the case of a finite time-horizon. Similarly Salminen & Vallois (2007) derived the joint distribution of maximum increase and decrease for the Brownian motion. Landriault et al. (2015) developed formulas for the frequency of drawdowns for the Brownian motion (by computing the Laplace transform).

Mijatovic & Pistorius (2012) and Landriault et al. (2017) derived formulas for the drawdown for the case of Lévy processes. Zhang & Hadjiliadis (2012) developed formulas that provide for the drawdown and, in addition the drawdown time (time in which the drawdown plays out). Both measures are important for analyzing crashes, as crashes are characterized by a high speed of market tumble. Hayes (2006) modeled the market as a Markov chain, and derived formulas for the drawdown, and the probability of recovering from a drawdown, assuming a correlated process.

In addition to pure mathematical analysis, some works studied the relation of maximum drawdown to financial and other related instruments and applications. For example, Meilijson (2003) studied a Brownian motion when certain optimal stopping is applied, with its implications on maximum drawdown. Carr et al. (2011) considered the maximum drawdown as a risk insurance (against market declines). Landriault et al. (2017) analyzed drawdown for the case of insurance risk. In it they considered the random nature of insurance claims' arrivals. Vecer (2007) proposed the use of options based on maximum drawdown to provide risk protection instruments. Rotundo & Navarra (2007) and Petroni & Rotundo (2008) studied the relation between maximum drawdown and stock market crashes. Goldberg & Mahmoud (2014) derived the average of worst case maximum drawdowns exceeding a quantile of the maximum drawdown distribution. In follow up work (Goldberg & Mahmoud, 2017) they propose this as a new risk measure. Mahmoud (2015) proposed a risk measure that incorporates the speed of declines associated with drawdowns.

One can observe that most of the work focused on continuous time stochastic processes. As far as we know, the work proposed here is the first one to produce a complete solution for the MDD probability density of the discrete stochastic process case.

3 Preliminaries

Let x_t be a random process characterized by

$$x_t = x_{t-1} + u, \tag{1}$$

where u is a random increment with density p_u. We assume that successive increments are independent. Let the starting point be $x_0 = 0$.

Define the running maximum as $y_t = \max_{\tau \in \{1,\dots,t\}} x_\tau$. The maximum drawdown (MDD) is given by

$$z_t = \max_{\tau \in \{1,\dots,t\}} \Big[y_\tau - x_\tau \Big]. \tag{2}$$

Figure 1 illustrates the concept of the maximum drawdown. The following three equations define the formulation of the MDD problem:

$$x_t = x_{t-1} + u, \qquad u \sim p_u \tag{3}$$

$$y_t = \max\big(y_{t-1}, x_t\big) \tag{4}$$

$$z_t = \max\big(z_{t-1}, y_{t-1} - x_t\big) \tag{5}$$

with $x_0 = 0, \quad y_0 = 0, \quad z_0 = 0$. The reason for y_{t-1} instead of y_t in Eq. 5 is that the maximum is necessarily achieved at a time less than t if $y_t - x_t$ has a positive value.

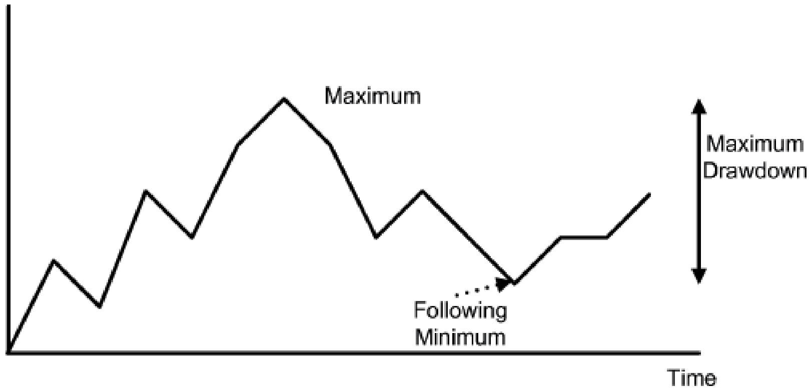

Figure 1. An illustration of the definition of the maximum drawdown

4 Derivation of MDD Probability Density

Using the Chapman-Kolmogorov equations, we obtain

$$\begin{aligned}
& p(x_t, y_t, z_t) \\
= & \int_0^\infty \int_0^\infty \int_{-\infty}^\infty p(x_t, y_t, z_t | x_{t-1}, y_{t-1}, z_{t-1}) p(x_{t-1}, y_{t-1}, z_{t-1}) dx_{t-1} dy_{t-1} dz_{t-1} \\
= & \int_0^\infty \int_0^\infty \int_{-\infty}^\infty p(y_t, z_t | x_t, y_{t-1}, z_{t-1}) p(x_t | x_{t-1}, y_{t-1}, z_{t-1}) \\
& \qquad p(x_{t-1}, y_{t-1}, z_{t-1}) dx_{t-1} dy_{t-1} dz_{t-1} \\
= & \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \delta\big(y_t - \max(y_{t-1}, x_t)\big) \delta\big(z_t - \max(z_{t-1}, y_{t-1} - x_t)\big) p_u(x_t - x_{t-1}) \\
& \qquad p(x_{t-1}, y_{t-1}, z_{t-1}) dx_{t-1} dy_{t-1} dz_{t-1},
\end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta function and we used the Markov property in the first term of the second equation and that allowed us to remove the conditioning on x_{t-1}. We used also the fact that

$$p(x_t | x_{t-1}, y_{t-1}, z_{t-1}) = p(x_t | x_{t-1}) = p_u(x_t - x_{t-1}) \tag{6}$$

and used Eqs (4) and (5) to evaluate $p(y_t, z_t | x_t, y_{t-1}, z_{t-1})$.

To evaluate the integral we consider a number of possible ranges for the variables. In the first case the process makes "new highs". This means that the new maximum equals the final process value: $y_t = x_t$. Because, for this case $y_{t-1} \le y_t = x_t$, we can write $\delta\Big(y_t - \max(y_{t-1}, x_t)\Big) = \delta(y_t - x_t)$. Also, $\delta\Big(z_t - \max(z_{t-1}, y_{t-1} - x_t)\Big) = \delta(z_t - z_{t-1})$ because $y_{t-1} - x_t \le 0$. We get

$$\begin{aligned}
p(x_t, y_t, z_t) &= \delta(y_t - x_t) \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \delta(z_t - z_{t-1}) p_u(x_t - x_{t-1}) \\
& \qquad p(x_{t-1}, y_{t-1}, z_{t-1}) dx_{t-1} dy_{t-1} dz_{t-1} \\
&= \delta(y_t - x_t) \int_0^{x_t} \int_{y_{t-1} - z_t}^{y_{t-1}} p_u(x_t - x_{t-1}) p(x_{t-1}, y_{t-1}, z_t) dx_{t-1} dy_{t-1}.
\end{aligned}$$

Note that the change in integration limits reflects the fact that for this particular case $0 \leq y_{t-1} \leq x_t$, and $y_{t-1} - z_t \leq x_{t-1} \leq y_{t-1}$. The latter inequality is due to the fact that $z_t = z_{t-1}$ (no new drawdown at t), $z_{t-1} \geq y_{t-1} - x_{t-1}$ (by definition), and $x_{t-1} \leq y_{t-1}$ (by definition).

The next case to consider is when $y_t > x_t$. For such case the argument inside the δ in $\delta\big(y_t - \max(y_{t-1}, x_t)\big)$ equals zero only when $y_t = y_{t-1}$. So we substitute $\delta(y_t - y_{t-1})$ in place of $\delta\big(y_t - \max(y_{t-1}, x_t)\big)$. We partition the integral w.r.t. z_{t-1} into two ranges: from 0 to $y_t - x_t$, and from $y_t - x_t$ to ∞. In the first range $\delta\big(z_t - \max(z_{t-1}, y_{t-1} - x_t)\big) = \delta(z_t - y_{t-1} + x_t) = \delta(z_t - y_t + x_t)$ while in the second range $\delta\big(z_t - \max(z_{t-1}, y_{t-1} - x_t)\big) = \delta(z_t - z_{t-1})$. We get

$$\begin{aligned} p(x_t, y_t, z_t) &= \int_0^{y_t - x_t} \int_0^{\infty} \int_{-\infty}^{\infty} \delta\Big(y_t - \max(y_{t-1}, x_t)\Big) \delta(z_t - y_t + x_t) p_u(x_t - x_{t-1}) \\ &\qquad p(x_{t-1}, y_{t-1}, z_{t-1}) dx_{t-1} dy_{t-1} dz_{t-1} \\ &\quad + \int_{y_t - x_t}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \delta\Big(y_t - \max(y_{t-1}, x_t)\Big) \delta(z_t - z_{t-1}) p_u(x_t - x_{t-1}) \\ &\qquad p(x_{t-1}, y_{t-1}, z_{t-1}) dx_{t-1} dy_{t-1} dz_{t-1} \\ &= \delta(z_t - y_t + x_t) \int_0^{y_t - x_t} \int_{y_t - z_{t-1}}^{y_t} p_u(x_t - x_{t-1}) p(x_{t-1}, y_t, z_{t-1}) dx_{t-1} dz_{t-1} \\ &\quad + \int_{y_t - z_t}^{y_t} p_u(x_t - x_{t-1}) p(x_{t-1}, y_t, z_t) dx_{t-1}. \end{aligned}$$

The limits of the first integral w.r.t. x_{t-1} are as given above because $x_{t-1} \geq y_{t-1} - z_{t-1} = y_t - z_{t-1}$ (the latter is due the argument given above that $y_t = y_{t-1}$ in this case), and because $x_{t-1} \leq y_{t-1} = y_t$. Combining all previous equations, we get the final equation:

$$\begin{aligned} p(x_t, y_t, z_t) &= \mathbf{1}(y_t - x_t \geq 0) \mathbf{1}(z_t - y_t + x_t \geq 0) \Bigg[\delta(z_t - y_t + x_t) \cdot \\ &\quad \int_0^{y_t - x_t} \int_{y_t - z_{t-1}}^{y_t} p_u(x_t - x_{t-1}) p(x_{t-1}, y_t, z_{t-1}) dx_{t-1} dz_{t-1} \\ &\quad + \int_{y_t - z_t}^{y_t} p_u(x_t - x_{t-1}) p(x_{t-1}, y_t, z_t) dx_{t-1} \\ &\quad \delta(y_t - x_t) \int_0^{x_t} \int_{y_{t-1} - z_t}^{y_{t-1}} p_u(x_t - x_{t-1}) p(x_{t-1}, y_{t-1}, z_t) dx_{t-1} dy_{t-1} \Bigg], \end{aligned}$$

where $\mathbf{1}$ is the indicator function (it equals 1 if the argument is true, and zero otherwise). The existence of these functions is to enforce the fact that outside the ranges $z_t - y_t + x_t \geq 0$ or $y_t \geq x_t$ the density is zero because otherwise the definitions of the variables would be violated. We apply these equations to recursively obtain the joint density. First, we obtain the initial density for $t = 1$. For such case, there are two possibilities: $x_1 \geq 0$ and $x_1 < 0$, each of which gives a specific expression for the density, leading to:

$$p(x_1, y_1, z_1) = p_u(x_1) \delta(y_1 - x_1) \delta(z_1) \mathbf{1}(x_1 \geq 0) + p_u(x_1) \delta(z_1 + x_1) \delta(y_1) \mathbf{1}(x_1 < 0). \tag{7}$$

Moving to $t = 2$ and applying the density update equations, we get

$$\begin{aligned} p(x_2, y_2, z_2) &= \delta(y_2 - x_2) \delta(z_2) \mathbf{1}(x_2 \geq 0) \int_0^{x_2} p_u(x_2 - y_1) p_u(y_1) dy_1 \\ &\quad + \delta(y_2 - x_2) \mathbf{1}(x_2 \geq 0) p_u(x_2 + z_2) p_u(-z_2) \\ &\quad + \delta(x_2 + z_2) \delta(y_2) \mathbf{1}(x_2 < 0) \int_0^{-x_2} p_u(x_2 + z_1) p_u(-z_1) dz_1 \\ &\quad + \delta(z_2 - y_2 + x_2) \mathbf{1}(y_2 \geq x_2) p_u(x_2 - y_2) p_u(y_2) \\ &\quad + \delta(y_2) \mathbf{1}(x_2 < 0) \mathbf{1}(z_2 \geq -x_2) p_u(x_2 + z_2) p_u(-z_2). \end{aligned}$$

By observing the previous equation, and generalizing to the case of general t, we find that there are six possible terms in the expression for $p(x_t, y_t, z_t)$:

- First term $= \delta(y_t - x_t)\delta(z_t)\mathbf{1}(x_t \geq 0)a_t(x_t)$, for some function $a_t(x_t)$. This corresponds to the situation where the process has gone up in every single time step till time t, leading to $y_t = x_t$ and zero drawdown. Figure 2 depicts this situation.

- Second term $= \delta(y_t - x_t)\mathbf{1}(x_t \geq 0)b_t(x_t, z_t)$ for some function $b_t(x_t, z_t)$. This corresponds to the situation where the process is making new highs (i.e. $y_t = x_t$), but has incurred some drawdown at a previous time step. Figure 3 shows an example of this case.

- Third term $= \delta(x_t + z_t)\delta(y_t)\mathbf{1}(x_t < 0)c_t(x_t)$, for some function $c_t(x_t)$. This corresponds to the situation where the process has never gone above the zero level (leading to $y_t = 0$). In addition, x_t is the lowest point and is therefore defining a new level for the maximum drawdown. See Figure 4 for an illustration.

- Fourth term $= \delta(z_t - y_t + x_t)\mathbf{1}(y_t \geq x_t)d_t(x_t, y_t)$, for some function $d_t(x_t, y_t)$. This corresponds to the situation where the process x_t is defining a new level for the maximum drawdown. See Figure 5 for an illustration.

- Fifth term $= \delta(y_t)\mathbf{1}(x_t < 0)\mathbf{1}(z_t \geq -x_t)e_t(x_t, z_t)$ for some function $e_t(x_t, z_t)$. This corresponds to the situation where the process has never gone above the zero level (leading to $y_t = 0$). However, unlike the third term, x_t is not the lowest point and is therefore not defining a new level for the maximum drawdown. See Figure 6 for an illustration.

- Sixth term $= \mathbf{1}(z_t \geq y_t - x_t)\mathbf{1}(y_t \geq x_t)f_t(x_t, y_t, z_t)$, for some function $f_t(x_t, y_t, z_t)$. This corresponds to the most general case. In this situation, neither $y_t = 0$, nor is x_t defining a new level for the drawdown. Figure 7 shows an example of this situation. This particular term is the only one that does not appear in the case study of $t = 2$ above. However, it will appear in subsequent time steps.

Thus, assuming that

$$\begin{aligned}
p(x_{t-1}, y_{t-1}, z_{t-1}) \quad = \quad & \delta(y_{t-1} - x_{t-1})\delta(z_{t-1})\mathbf{1}(x_{t-1} \geq 0)a_{t-1}(x_{t-1}) \\
& +\delta(y_{t-1} - x_{t-1})\mathbf{1}(x_{t-1} \geq 0)b_{t-1}(x_{t-1}, z_{t-1}) \\
& +\delta(x_{t-1} + z_{t-1})\delta(y_{t-1})\mathbf{1}(x_{t-1} < 0)c_{t-1}(x_{t-1}) \\
& +\delta(z_{t-1} - y_{t-1} + x_{t-1})\mathbf{1}(y_{t-1} \geq x_{t-1})d_{t-1}(x_{t-1}, y_{t-1}) \\
& +\delta(y_{t-1})\mathbf{1}(x_{t-1} < 0)\mathbf{1}(z_{t-1} \geq -x_{t-1})e_{t-1}(x_{t-1}, z_{t-1}) \\
& +\mathbf{1}(z_{t-1} \geq y_{t-1} - x_{t-1})\mathbf{1}(y_{t-1} \geq x_{t-1})f_{t-1}(x_{t-1}, y_{t-1}, z_{t-1})
\end{aligned}$$

and applying the integral equation, we obtain:

$$\begin{aligned}
p(x_t, y_t, z_t) \quad = \quad & \delta(y_t - x_t)\delta(z_t)\mathbf{1}(x_t \geq 0)a_t(x_t) + \delta(y_t - x_t)\mathbf{1}(x_t \geq 0)b_t(x_t, z_t) \\
& +\delta(x_t + z_t)\delta(y_t)\mathbf{1}(x_t < 0)c_t(x_t) + \delta(z_t - y_t + x_t)\mathbf{1}(y_t \geq x_t)d_t(x_t, y_t) \\
& +\delta(y_t)\mathbf{1}(x_t < 0)\mathbf{1}(z_t \geq -x_t)e_t(x_t, z_t) \\
& +\mathbf{1}(z_t \geq y_t - x_t)\mathbf{1}(y_t \geq x_t)f_t(x_t, y_t, z_t),
\end{aligned}$$

where

$$a_t(x_t) = \int_0^{x_t} p_u(x_t - y_{t-1})a_{t-1}(y_{t-1})dy_{t-1}, \tag{8}$$

$$\begin{aligned}
b_t(x_t, z_t) &= \int_0^{x_t} p_u(x_t - y_{t-1}) b_{t-1}(y_{t-1}, z_t) dy_{t-1} + p_u(x_t + z_t) c_{t-1}(-z_t) \\
&+ \int_0^{x_t} p_u(x_t + z_t - y_{t-1}) d_{t-1}(y_{t-1} - z_t, y_{t-1}) dy_{t-1} \\
&+ \int_{-z_t}^{0} p_u(x_t - x_{t-1}) e_{t-1}(x_{t-1}, z_t) dx_{t-1} \\
&+ \int_0^{x_t} \int_{y_{t-1} - z_t}^{y_{t-1}} p_u(x_t - x_{t-1}) f_{t-1}(x_{t-1}, y_{t-1}, z_t) dx_{t-1} dy_{t-1},
\end{aligned}$$

$$\begin{aligned}
c_t(x_t) &= \int_0^{-x_t} \int_{-z_{t-1}}^{0} p_u(x_t - x_{t-1}) e_{t-1}(x_{t-1}, z_{t-1}) dx_{t-1} dz_{t-1} \\
&+ \int_0^{-x_t} p_u(x_t + z_{t-1}) c_{t-1}(-z_{t-1}) dz_{t-1},
\end{aligned}$$

$$\begin{aligned}
d_t(x_t, y_t) &= p_u(x_t - y_t) a_{t-1}(y_t) + p_u(x_t - y_t) \int_0^{y_t - x_t} b_{t-1}(y_t, z_{t-1}) dz_{t-1} \\
&+ \int_0^{y_t - x_t} p_u(x_t - y_t + z_{t-1}) d_{t-1}(y_t - z_{t-1}, y_t) dz_{t-1} \\
&+ \int_0^{y_t - x_t} \int_{y_t - z_{t-1}}^{y_t} p_u(x_t - x_{t-1}) f_{t-1}(x_{t-1}, y_t, z_{t-1}) dx_{t-1} dz_{t-1},
\end{aligned}$$

$$\begin{aligned}
e_t(x_t, z_t) &= p_u(x_t + z_t) c_{t-1}(-z_t) \\
&+ \int_{-z_t}^{0} p_u(x_t - x_{t-1}) e_{t-1}(x_{t-1}, z_t) dx_{t-1},
\end{aligned}$$

$$\begin{aligned}
f_t(x_t, y_t, z_t) &= p_u(x_t - y_t) b_{t-1}(y_t, z_t) \\
&+ p_u(x_t - y_t + z_t) d_{t-1}(y_t - z_t, y_t) \\
&+ \int_{y_t - z_t}^{y_t} p_u(x_t - x_{t-1}) f_{t-1}(x_{t-1}, y_t, z_t) dx_{t-1}.
\end{aligned}$$

The starting point is given from Eq. (7): $a_1(x_1) = p_u(x_1)$, $b_1(x_1, z_1) = 0$, $c_1(x_1) = p_u(x_1)$, $d_1(x_1, y_1) = 0$, $e_1(x_1, z_1) = 0$, and $f_1(x_1, y_1, z_1) = 0$. Once we obtain the joint density $p(x_t, y_t, z_t)$, the marginal density of the maximum drawdown can be obtained in a straightforward way, as follows:

$$p(z_t) = \int_0^{\infty} \int_{-\infty}^{\infty} p(x_t, y_t, z_t) dx_t dy_t. \tag{9}$$

Applying this formula gives the following final probability density function of the maximum drawdown:

$$\begin{aligned}
p(z_t) = \delta(z_t) \int_0^{\infty} a_t(x_t) dx_t + \int_0^{\infty} b_t(x_t, z_t) dx_t + c_t(-z_t) + \int_0^{\infty} d_t(y_t - z_t, y_t) dy_t \\
+ \int_{-\infty}^{0} e_t(x_t, z_t) dx_t + \int_0^{\infty} \int_{y_t - z_t}^{y_t} f_t(x_t, y_t, z_t) dx_t dy_t.
\end{aligned} \tag{10}$$

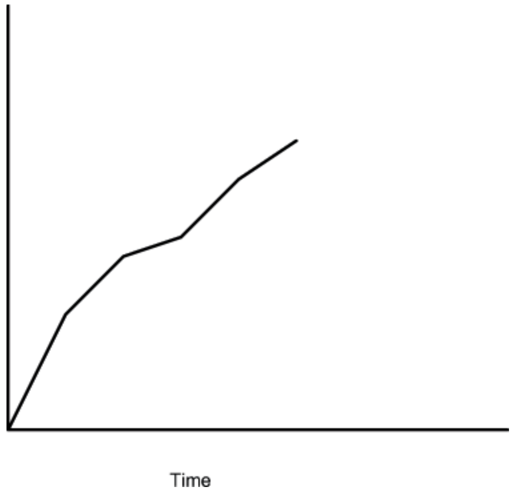

Figure 2. An example of the process that yields Term 1 (no drawdowns, always new highs, and $y_t = x_t$)

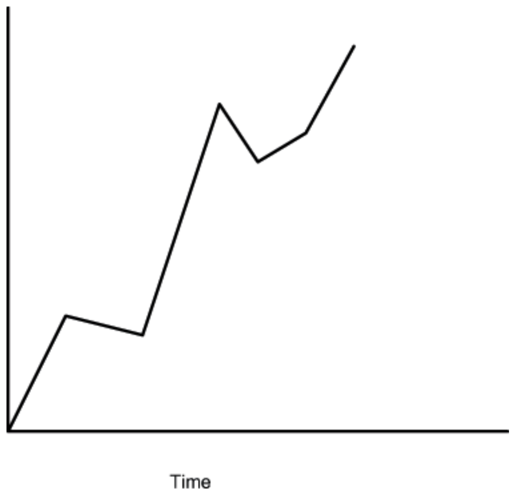

Figure 3. An example of the process that yields Term 2 (a new high, i.e. $y_t = x_t$, but some drawdowns)

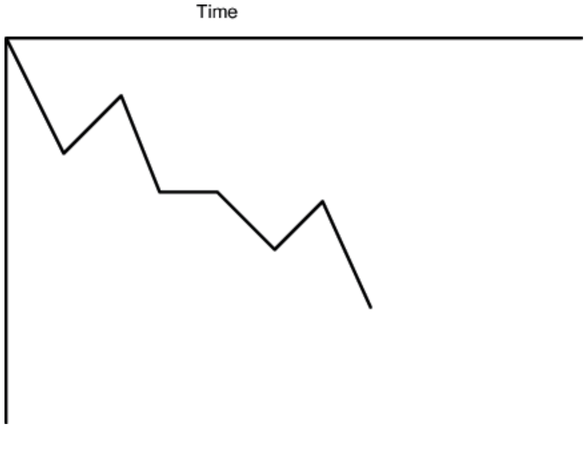

Figure 4. An example of the process that yields Term 3 (a new MDD at time t, and zero y_t, i.e. all process values below starting point)

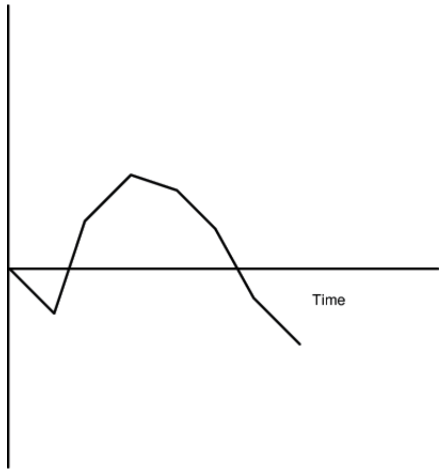

Figure 5. An example of the process that yields Term 4 (a new MDD at time t, but a positive y_t)

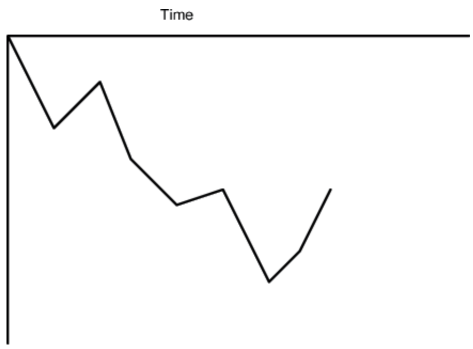

Figure 6. An example of the process that yields Term 5 (zero y_t, but no new MDD at time t)

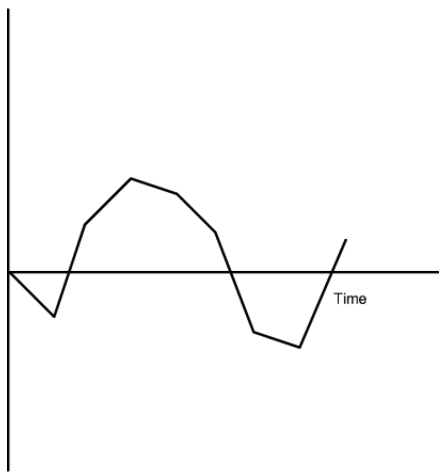

Figure 7. An example of the process that yields Term 6
(a general case, no new high and no new MDD at time t)

5 Implementation Examples

To illustrate the derived formulas we have considered a simple example of normal increments ($p_u \equiv N(0,1)$, $T = 4$). We used simple numerical integration to compute the recursive integrals in the formulas. Figure 8 shows the probability density function, as derived by the formulas, versus a simulation computation. This confirms the accuracy of the derived formula.

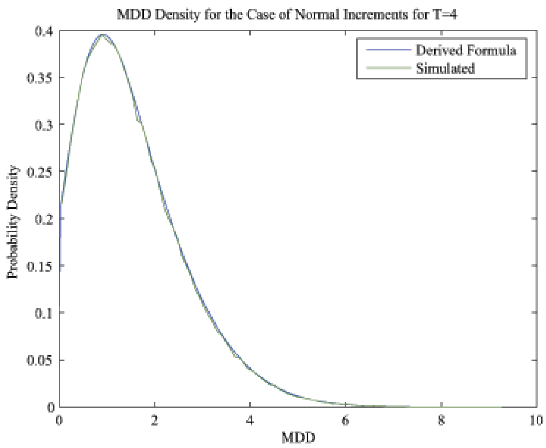

Figure 8. MDD density for the case of normal increments for T=4

In the next set of experiments we explore the MDD density for some processes. First we considered a normal process, with increment density $N(0,1)$, ran for 12 steps forward ($T = 12$, akin to an investment horizon of 12 months). Figure 9 shows the evolution of the MDD density every two time steps, until reaching the final time step of $T = 12$. As expected, the MDD density spreads out with time, but this slows down as T increases. In another experiment we considered a process with double exponential exponential increment density, given by:

$$p_u(u) = \frac{1}{2b} e^{-\left[\frac{|u-\mu|}{b}\right]}, \tag{11}$$

where the parameters are selected as $\mu = 0$ $b = \frac{1}{\sqrt{2}}$ such that this double exponential process has the same increment mean and standard deviation as the normal process implemented above. Figure 10 shows the evolution of the MDD density every two time steps, until reaching the final time step of $T = 12$. One can observe that the MDD density is very close to that of the normal process. This suggests that it is the mean and standard deviation of the increments that play the major role in determining the MDD density's shape, rather than tail behavior.

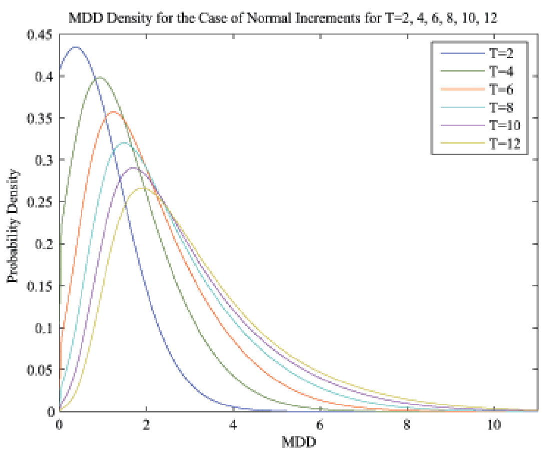

Figure 9. The evolution of the MDD density for the case of normal increments for various values of T (T=2, 4, 6, 8, 10, 12). Note that there is a small delta function at $z = 0$, not shown in the figure, that progressively becomes smaller an smaller as T increases.

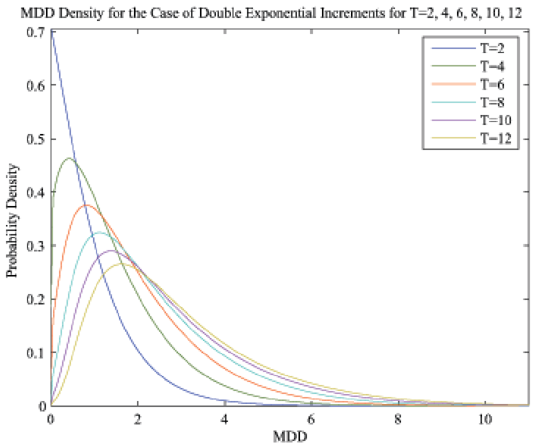

Figure 10. The evolution of the MDD density for the case of Double exponential increments for various values of T (T=2, 4, 6, 8, 10, 12). Note that there is a small delta function at $z = 0$, not shown in the figure, that progressively becomes smaller an smaller as T increases.

6 Conclusion

In this work we have derived recursion integral equations for the maximum drawdown density problem for discrete processes. The advantage of the proposed work is that it deals with discrete processes, a case rarely considered in the literature. In real financial markets time is discrete in actuality. The other contribution is that it gives a complete characterization of the density, rather than the moments as in most other works. The benefit of this study is that it could open the door for further analysis and future studies for discrete processes. For example, one could consider asymptotic $T \longrightarrow \infty$ behavior, or one could consider tail behavior of the MDD density (one could consider the general behavior of these densities asymptotically and in the tail given general properties of p_u, such as bounded mean and variance).

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